Class Ex. #1

Consider the following functions.

$a(x) = \dfrac{x-4}{x+4}$ $\quad b(x) = \dfrac{x^2-4}{x^2+4}$ $\quad c(x) = \dfrac{x+4}{x^2-4}$ $\quad d(x) = \dfrac{x+2}{x^2-4}$ $\quad e(x) = \dfrac{x^2+4}{x+4}$

$c(x) = \dfrac{x+4}{(x-2)(x+2)}$

$d(x) = \dfrac{x+2}{(x-2)(x+2)} = \dfrac{1}{x-2},\ x \neq -2$

$f(x) = \dfrac{x^2+4}{x^2-4}$ $\quad g(x) = \dfrac{-x-4}{-x^2-4}$ $\quad h(x) = \dfrac{-x^2-4}{-x^2+4}$ $\quad i(x) = \dfrac{x^2-4}{x+2}$ $\quad j(x) = \dfrac{x^3-4x^2}{x^2-2x}$

$f(x) = \dfrac{x^2+4}{(x-2)(x+2)}$

$g(x) = \dfrac{-(x+4)}{-(x^2+4)} = \dfrac{x+4}{x^2+4}$

$h(x) = \dfrac{-(x^2+4)}{-(x^2-4)} = \dfrac{x^2+4}{(x-2)(x+2)}$

$i(x) = \dfrac{(x-2)(x+2)}{x+2} = x-2,\ x \neq -2$

$j(x) = \dfrac{x^2(x-4)}{x(x-2)} = \dfrac{x(x-4)}{x-2},\ x \neq$

Without sketching the graph of the function, determine which functions have

a) no discontinuities $\quad$ b, g

b) no vertical asymptote $\quad$ b, g, i

c) no horizontal asymptote $\quad$ e, i, j

d) the x-axis as a horizontal asymptote $\quad$ c, d, g

e) a horizontal asymptote with equation $y = 1$ $\quad$ a, b, f, h

f) point(s) of discontinuity $\quad$ d, i, j

Class Ex. #2

Algebraically determine the equations of any asymptotes and the coordinates of any point(s) of discontinuity on the graph of the function $f(x) = \dfrac{(2x-1)(x-5)}{x^2-2x-15}$.

State the domain and range of f.

$f(x) = \dfrac{(2x-1)(x-5)}{(x+3)(x-5)} = \dfrac{2x^2-11x+5}{x^2-2x-15}$

$= \dfrac{2x-1}{x+3},\ x \neq 5$

when $x = 5$, $\dfrac{2x-1}{x+3} = \dfrac{2(5)-1}{5+3} = \dfrac{9}{8}$

vertical asymptote $\quad x = -3$

horizontal asymptote $\quad y = 2$

point of discontinuity $\quad \left(5, \frac{9}{8}\right)$

domain $\quad \{x \mid x \neq -3, 5,\ x \in R\}$

range $\quad \{y \mid y \neq \frac{9}{8}, 2,\ y \in R\}$

Complete Assignment Questions #1 - #5

Assignment

1. For each of the graphs of the following rational functions, algebraically determine the equation of any asymptotes and the coordinates of any points of discontinuity.

a) $f(x) = \dfrac{2(x+1)}{x^2+1} = \dfrac{2x+2}{x^2+1}$

horizontal asymptote $y = 0$

b) $g(x) = -\dfrac{4x^3}{x^3+1}$ $x^3 + 1 = 0$, $x = -1$

horizontal asymptote $y = -4$

vertical asymptote $x = -1$

c) $h(x) = \dfrac{3-x^4}{3x^4+6}$

horizontal asymptote $y = -\frac{1}{3}$

d) $k(x) = \dfrac{x+4}{x^2+5x+4} = \dfrac{x+4}{(x+1)(x+4)} = \dfrac{1}{x+1}$ $x \neq -4$

$\left(\dfrac{1}{-4+1} = -\dfrac{1}{3}\right)$

horizontal asymptote $y = 0$

vertical asymptote $x = -1$

point of discontinuity $\left(-4, -\frac{1}{3}\right)$

e) $f(x) = \dfrac{2(3x-1)(x+4)}{3x^2+4x+1}$

$3x^2+4x+1$
$= 3x^2+3x+x+1$
$= 3x(x+1)+1(x+1)$
$= (x+1)(3x+1)$

$2(3x-1)(x+4)$
$= 2(3x^2+11x-4)$
$= 6x^2+22x-8$

$f(x) = \dfrac{2(3x-1)(x+4)}{(x+1)(3x+1)}$ or $\dfrac{6x^2+22x-8}{3x^2+4x+1}$

horizontal asymptote $y = 2$

vertical asymptotes $x = -1$, $x = -\frac{1}{3}$

f) $f(x) = \dfrac{2(3x-1)(x+4)}{3x^2+10x-8}$

$3x^2+10x-8$
$= 3x^2-2x+12x-8$
$= x(3x-2)+4(3x-2)$
$= (3x-2)(x+4)$

$f(x) = \dfrac{2(3x-1)(x+4)}{(3x-2)(x+4)} = \dfrac{2(3x-1)}{3x-2}$ $x \neq -4$

or $f(x) = \dfrac{6x^2+22x-8}{3x^2+10x-8}$

If $x = -4$, $\dfrac{2(3x-1)}{3x-2} = \dfrac{2(-13)}{-14} = \dfrac{13}{7}$

horizontal asymptote $y = 2$

vertical asymptote $x = \frac{2}{3}$

point of discontinuity $\left(-4, \frac{13}{7}\right)$

2. Consider the functions $f(x) = x^2 - 9$ and $g(x) = 9x - x^3$.

a) Express $f(x)$ and $g(x)$ in factored form.

$f(x) = x^2 - 9 = (x-3)(x+3)$

$g(x) = 9x - x^3 = x(9 - x^2) = x(3-x)(3+x)$

b) Express the function $\left(\frac{f}{g}\right)(x)$ in simplest form.

$\left(\frac{f}{g}\right)(x) = \frac{x^2-9}{9x-x^3} = \frac{(x-3)(x+3)}{x(3-x)(3+x)} = -\frac{1}{x}, \quad x \neq \pm 3$

c) Algebraically determine the equation of any asymptotes and the coordinates of any points of discontinuity on the graph of $y = \left(\frac{f}{g}\right)(x)$.

If $x = -3$, $-\frac{1}{x} = \frac{-1}{-3} = \frac{1}{3}$

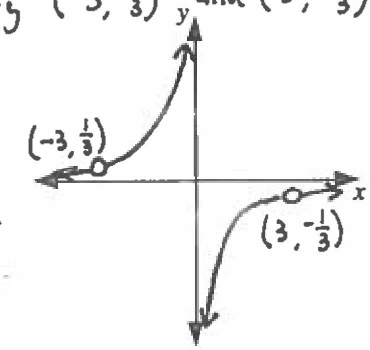
If $x = 3$, $-\frac{1}{x} = -\frac{1}{3}$

horizontal asymptote $y = 0$

vertical asymptote $x = 0$

points of discontinuity $(-3, \frac{1}{3})$ and $(3, -\frac{1}{3})$

d) Sketch a graph of $y = \left(\frac{f}{g}\right)(x)$ on the grid illustrating the features in c).



e) Determine the domain and range of the function $\left(\frac{f}{g}\right)(x)$.

domain $\{x \mid x \neq 0, \pm 3, x \in R\}$

range $\{y \mid y \neq 0, \pm \frac{1}{3}, y \in R\}$

3. Consider the functions $f(x) = \frac{3}{x+2}$ and $g(x) = x - 2$.

a) Show that $(g \circ f)(x) = \frac{-2x-1}{x+2}$.

$(g \circ f)(x) = g\left(\frac{3}{x+2}\right) = \frac{3}{x+2} - 2 = \frac{3}{x+2} - \frac{2(x+2)}{x+2} = \frac{3-2(x+2)}{x+2}$

$= \frac{3-2x-4}{x+2} = \frac{-2x-1}{x+2}$

b) Determine the equations of the asymptotes of the graph of $y = (g \circ f)(x)$.

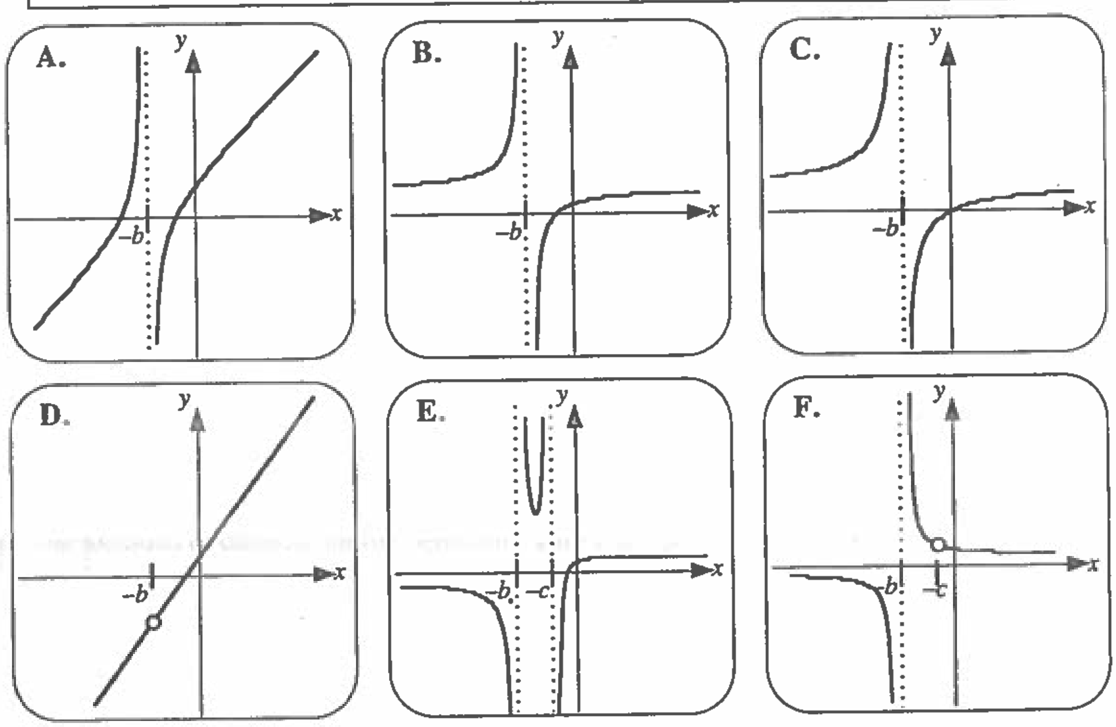
horizontal asymptote $y = -2$ vertical asymptote $x = -2$

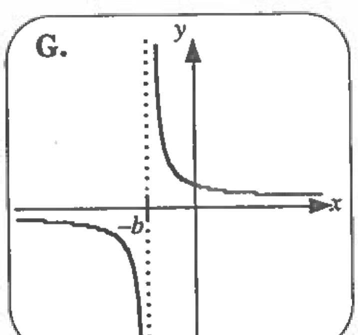
c) State the domain and range of the function $(g \circ f)(x)$.

domain $\{x \mid x \neq -2, x \in R\}$ range $\{y \mid y \neq -2, y \in R\}$

4. The equations of seven rational functions and the graphs of these functions are shown. If a, b, and c are distinct natural numbers, match the set of rational functions to their graphs and explain the reasoning.

Function 1	**Function 2**	**Function 3**	**Function 4**
$y=\dfrac{1}{x+b}$	$y=\dfrac{x}{x+b}$	$y=\dfrac{x+a}{x+b}$	$y=\dfrac{(x+a)(x+c)}{x+b}$
Function 5	**Function 6**	**Function 7**	
$y=\dfrac{(x+a)(x+b)}{x+b}$	$y=\dfrac{x+a}{(x+b)(x+c)}$	$y=\dfrac{x+c}{(x+b)(x+c)}$	





Function 1 has a vertical asymptote $x=-b$ and a horizontal asymptote $y=0$, so $1\Rightarrow G$.
Functions 2 and 3 have a vertical asymptote $x=-b$ and a horizontal asymptote $y=1$. Function 2 passes through the origin and Function 3 does not, so $2\Rightarrow C$ and $3\Rightarrow B$.
Functions 4 and 5 have no horizontal asymptotes. Function 4 has a vertical asymptote $x=-b$ and Function 5 has a point of discontinuity at $x=-b$, so $4\Rightarrow A$ and $5\Rightarrow D$.
Functions 6 and 7 have two discontinuities. Function 6 has two vertical asymptotes, so $6\Rightarrow E$. Function 7 has one vertical asymptote and one point of discontinuity, so $7\Rightarrow F$.

$1\Rightarrow G$ $5\Rightarrow D$
$2\Rightarrow C$ $6\Rightarrow E$
$3\Rightarrow B$ $7\Rightarrow F$
$4\Rightarrow A$

Multiple Choice

5. Which of the following functions has a point of discontinuity at (p, q)?

A. $y = \dfrac{(x-p)(x-q)}{x-p}$ $= x - q,\ x \neq p$ If $x = p,\ y = p - q$

B. $y = \dfrac{(x-p)(x-q)}{x-q}$ $= x - p,\ x \neq q$

C. $y = \dfrac{(x-p)(x-p+q)}{x-p}$ $= x - p + q,\ x \neq p$ If $x = p,\ y = p - p + q = q$

D. $y = \dfrac{(x-q)(x+p-q)}{x-q}$ $= x + p - q,\ x \neq q$

Answer Key

1. **a)** horizontal asymptote $y = 0$. **b)** vertical asymptote $x = -1$, horizontal asymptote $y = -4$

c) horizontal asymptote $y = -\frac{1}{3}$

d) vertical asymptote $x = -1$, horizontal asymptote $y = 0$, point of discontinuity $\left(-4, -\frac{1}{3}\right)$.

e) vertical asymptote $x = -1, x = -\frac{1}{3}$, horizontal asymptote $y = 2$.

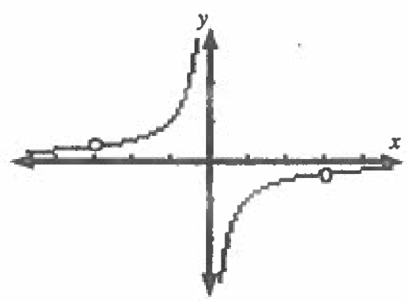
f) vertical asymptote $x = \frac{2}{3}$, horizontal asymptote $y = 2$, point of discontinuity $\left(-4, \frac{13}{7}\right)$.

2. **a)** $f(x) = (x-3)(x+3)$, $g(x) = x(3-x)(3+x)$

b) $\left(\dfrac{f}{g}\right)(x) = -\dfrac{1}{x}, x \neq \pm 3$

c) vertical asymptote $x = 0$, horizontal asymptote $y = 0$, points of discontinuity $\left(-3, \frac{1}{3}\right), \left(3, -\frac{1}{3}\right)$.

d) See graph at side.



e) Domain: $\{x \mid x \neq 0, \pm 3, x \in R\}$

Range: $\left\{y \mid y \neq 0, \pm\frac{1}{3}, y \in R\right\}$

3. **b)** vertical asymptote $x = -2$, horizontal asymptote $y = -2$

c) Domain: $\{x \mid x \neq -2, x \in R\}$ Range: $\left\{y \mid y \neq -2, y \in R\right\}$

4. Function 1 → G, Function 2 → C, Function 3 → B, Function 4 → A,
Function 5 → D, Function 6 → E, Function 7 → F.

- Function 1 has a vertical asymptote $x = -b$ and a horizontal asymptote $y = 0$, so Function 1 → G.
- Functions 2 and 3 have a vertical asymptote $x = -b$ and a horizontal asymptote $y = 1$. Function 2 passes through the origin and Function 3 does not, so Function 2 → C and Function 3 → B.
- Functions 4 and 5 have no horizontal asymptotes. Function 4 has a vertical asymptote $x = -b$, and Function 5 has a point of discontinuity at $x = -b$, so Function 4 → A and Function 5 → D.
- Functions 6 and 7 have two discontinuities. Function 6 has two vertical asymptotes, so Function 6 → E. Function 7 has one vertical asymptote and one point of discontinuity, so Function 7 → F.

5. C